A Note on Industrial Upgrading

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Abstract. Ju(2015) poses an interesting model for industrial upgrading, but they made some mistakes in solving their model. In this paper, two modifications of their results are made: (i) the relationship between the speed of industry upgrading and the productivity parameter depends on the intertemporal elasticity of substitution, if it is bigger than 2, then, this relationship is positive; if it is 2, then, the speed of industry upgrading is irrelevant to the productivity parameter; if it is smaller than 2, then, this relationship is negative; (ii) in general, each industry experiences three phases of development, rise, peak and descent, and follows a trapezoid-shaped pattern. In particular, the peak period could be quite long, and the rise and descent periods could be very short.

Keywords. Industry upgrading, Industry life-cycle, Non-smooth dynamic optimization.

Throughout this note, for simplicity of notation, for any variable, depending on time, we do not write out (t), e.g., we write x, instead of $x(t)$, and we use $\mathring{x} = \dot{x}/x$ to represent its growth rate.

1 Problem

First of all, let's restate the model in [1]. Consider an economy, in which there are two types of sectors: the first type consists of one industry producing the capital good, and the second type consists of a series of industries producing a series of intermediate goods and an industry the final consumption good.

The capital good is produced using an AK technology, its motion obeys

$$\dot{Z} = rZ - K,$$

where Z is the capital stock, and K is the working capital, which can be used in the productions of the intermediate goods, and $r > 0$ is a constant, the real interest rate of the capital.

The final consumption good C is produced by a series of intermediate goods $C_0, C_1, ..., C_n, ...$, which, in turn, are produced in industry 0, industry 1, ..., industry n,..., respectively.

The production function of the final good is

$$C = \sum_{n=0}^{\infty} b^n C_n,$$

where $b > 1$ is a constant.

As to the intermediate goods, the production functions are as follows respectively:

$$\begin{aligned} C_0 &= L_0, \\ C_n &= \min\left\{\frac{K_n}{a^n}, L_n\right\}, \quad n = 1, 2, ..., \end{aligned}$$

where K_n, L_n are capital and labor used in industry n respectively, and a is a constant, satisfying $a > b + 1$.

For simplicity, assume that the total labor endowment is normalized to 1. The total labor and total working capital are distributed among all the industries producing the intermediate goods, that is,

$$\begin{aligned} \sum_{n=1}^{\infty} K_n &= K, \\ \sum_{n=0}^{\infty} L_n &= 1. \end{aligned}$$

Suppose the utility function of the representative individual is

$$\int_0^{\infty} e^{-\rho t} C^{\alpha} dt,$$

where $\alpha \in (0, 1)$, and $\rho > 0$ is the social discount rate. We denote $\beta = 1 - \alpha$.

The social planner's goal is to maximize the representative individual's utility under all the constraints above, that is , his problem is the following dynamic optimization problem $\mathbb{P}$:

$$\begin{aligned} \max \quad & \int_0^{\infty} e^{-\rho t} C^{\alpha} dt, \\ \text{s.t.} \quad & \dot{Z} = rZ - K, \\ & C = \sum_{n=0}^{\infty} b^n C_n, \end{aligned}$$

$$
\begin{aligned}
&C_0 = L_0, \\
&C_n = \min\left\{\frac{K_n}{a^n}, L_n\right\}, \quad n = 1, 2, ..., \\
&\sum_{n=1}^{\infty} K_n = K, \\
&\sum_{n=0}^{\infty} L_n = 1, \\
&Z \geq 0, \quad K \geq 0, \\
&K_n \geq 0, \quad n = 1, 2, ..., \\
&L_n \geq 0, \quad n = 0, 1, 2, ..., \\
&Z(0) = Z_0,
\end{aligned}
$$

where $Z_0 > 0$ is given.

Assume $\rho < r < \rho/\alpha$. And for the use in the sequel, we denote $\delta = (r-\rho)/\beta$,

$$
\begin{aligned}
&a_0 = 0, \quad a_n = a^n, \quad n = 1, 2, ... \\
&k_n = \frac{b^{n+1} - b^n}{a_{n+1} - a_n}, \quad n = 0, 1, 2, ... \\
&\mathcal{A} = \{a_n, n = 0, 1, 2, ...\},
\end{aligned}
$$

and define a function π on $[0, \infty)$ as follows:

$$
\pi(K) = k_n(K - a_n) + b^n, \quad \text{if} \quad K \in [a_n, a_{n+1}), \quad n = 0, 1, 2, ...
$$

Clearly, $\{k_n\}_{n=0,1,2,...}$ is strictly decreasing and converging to 0, and π is continuous, concave, and piecewise linear.

It's easy to see that problem $\mathbb{P}$ can be solved in two steps. Firstly, we solve problem $\mathbb{P}_1$:

$$
\begin{aligned}
C \quad = \quad \max \quad & \sum_{n=0}^{\infty} b^n C_n, \\
\text{s.t.} \quad & C_0 = L_0, \\
& C_n = \min\left\{\frac{K_n}{a^n}, L_n\right\}, \quad n = 1, 2, ..., \\
& \sum_{n=1}^{\infty} K_n = K, \\
& \sum_{n=0}^{\infty} L_n = 1, \\
& K_n \geq 0, \quad n = 1, 2, ..., \\
& L_n \geq 0, \quad n = 0, 1, 2, ...,
\end{aligned}
$$

where $K \geq 0$ is given. We will see that its value function is just the function π defined above, then, $C = \pi(K)$.

Secondly, we solve problem $\mathbb{P}_2$:

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} \pi^\alpha(K) dt, \\ \text{s.t.} \quad & \dot{Z} = rZ - K, \\ & Z \geq 0, K \geq 0, \\ & Z(0) = Z_0, \end{aligned}$$

where $Z_0 > 0$ is given,

2 Solution

We solve $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively.

2.1 Problem $\mathbb{P}_1$

Clearly, $\mathbb{P}_1$ is equivalent to

$$\begin{aligned} \max \quad & \sum_{n=0}^\infty b^n L_n, \\ \text{s.t.} \quad & \sum_{n=1}^\infty a^n L_n = K, \\ & \sum_{n=0}^\infty L_n = 1, \\ & L_n \geq 0, \quad n = 0, 1, 2, ..., \end{aligned}$$

where $K \geq 0$ is given.

Since the objective functional and the constraints are all linear, and at any point, all the binding constraints are linearly independent, then, $\{L_n\}_{n=0,1,2,...}$ is optimal iff there exist Lagrange multipliers μ, η, $\{\theta_n\}_{n=0,1,2,...}$ such that for any $n = 0, 1, 2, ...,$

$$\begin{gathered} b_n - \mu - \eta a_n + \theta_n = 0, \\ \theta_n \geq 0, \quad L_n \geq 0, \quad \theta_n L_n = 0. \end{gathered}$$

If $L_{n_1} > 0, L_{n_3} > 0$ for some $n_1 < n_2 < n_3$, then,

$$\begin{aligned} b_{n_1} &= \mu + \eta a_{n_1}, \\ b_{n_2} &\leq \mu + \eta a_{n_2}, \\ b_{n_3} &= \mu + \eta a_{n_3}, \end{aligned}$$

implying

$$\frac{b_{n_3} - b_{n_2}}{a_{n_3} - a_{n_2}} \geq \frac{b_{n_2} - b_{n_1}}{a_{n_2} - a_{n_1}},$$

which contradicts to the fact

$$\frac{b_{n_3} - b_{n_2}}{a_{n_3} - a_{n_2}} < \frac{b_{n_2} - b_{n_1}}{a_{n_2} - a_{n_1}}.$$

It follows that there exists a unique n such that $L_n > 0$, $L_{n+1} \geq 0$, and $L_j = 0$ for all $j \neq n, n+1$, therefore,

$$\begin{aligned} L_n + L_{n+1} &= 1, \\ a_n L_n + a_{n+1} L_{n+1} &= K, \end{aligned}$$

which yields

$$\begin{aligned} L_n &= \frac{a_{n+1} - K}{a_{n+1} - a_n}, \\ L_{n+1} &= \frac{K - a_n}{a_{n+1} - a_n}. \end{aligned}$$

Consequently

$$K \in [a_n, a_{n+1}),$$

which determines the n uniquely. And, correspondingly, the final good C will be

$$C = b_n L_n + b_{n+1} L_{n+1} = \pi(K).$$

Remark 1. The above results implies that when $K \in (a_n, a_{n+1})$ for some $n = 0, 1, 2, ...$, then, there exist two and only two industries (industry n and industry $n+1$) to produce intermediate goods, and when $K = a_n$ for some $n = 0, 1, 2, ...$, then, there exists one and only one industry (industry n) to produce the intermediate good. And hence, along with the increasing of the working capital, the industries keep upgrading continuously as the following pattern:

phase 0: industry 0; ($K = 0$)
phase 1: industries 0 and 1; ($K \in I_0$)
phase 2: industry 1; ($K = a_1$)
phase 3: industries 1 and 2; ($K \in I_1$)
......
phase $2n$: industry n; ($K = a_n$)
phase $2n+1$: industry n and $n+1$; ($K \in I_n$)
......

Notice that the economy may start from some specific phase demonstrated above, which is determined by the initial capital stock Z_0.

By solving problem $\mathbb{P}_2$, we will see how the working capital is growing, and correspondingly, the concrete pattern of the industry upgrading, especially, what the starting phase is, and how long each phase will remain.

Remark 2. At any time point, since the consumer is only one person, then, to some sense, the social planner's problem is equivalent to the Walrasian equilibrium problem, that is, the maximal outcome of the final good can be obtained by the completely competitive market mechanism, put it in another way, the invisible hand, just as done in [1].

2.2 Problem $\mathbb{P}_2$

We need a Lemma at first.

Lemma. Let $F(x,\theta)$ be a continuous function on $\{(x,\theta)|x \geq 0, \theta > 0\}$. Suppose that for any $\theta > 0$, the optimization problem

$$\max_{x\geq 0} F(x,\theta)$$

has a unique solution, which is denoted as $\varphi(\theta)$. If φ is bounded on any bounded interval, then, φ is continuous.

Proof. Suppose a sequence $\theta_n \to \theta$, denote $x_n = \varphi(\theta_n)$, $x = \theta$, we need to prove $x_n \to x$. Since $\{\theta_n\}$ is bounded, then, $\{x_n\}$ is bounded, and hence, for any subsequence of it, denoted as $\{x_{n'}\}$, there is a further subsequence $\{x_{n''}\}$ which converges to some point, say, $y \geq 0$. Then,

$$F(x_{n''}, \theta_{n''}) \geq F(x, \theta_{n''}),$$

letting $n'' \to \infty$, by the continuity of F, we get

$$F(y,\theta) \geq F(x,\theta),$$

therefore, $y = x$. And hence, $x_n \to x$. The proof is completed.

Now, we try to solve the problem $\mathbb{P}_2$.

Obviously, Mangasarian sufficiency condition is satisfied, and hence, put the current-value Hamiltonian function

$$H = \pi^\alpha(K) + \lambda(rZ - K),$$

by Pontryagin maximum principle, a path (Z,K) is optimal iff there exists continuous and piecewise smooth λ such that

$$0 = H_K = \alpha\,(\pi(K))^{-\beta}\,\pi'(K) - \lambda, \quad \forall K \notin \mathcal{A}, \tag{1}$$

$$-\dot{\lambda} + \rho\lambda = H_Z = r\lambda, \tag{2}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)Z(t) = 0. \tag{3}$$

In the sequel, based on the above conditions, we try to get the optimal path (Z,K) and the corresponding $C = \pi(K)$.

Clearly, it always holds $C \geq 1$. And hence, by (1), we know $\lambda > 0$, therefore, by (2), we get

$$\mathring{\lambda} = \rho - r, \tag{4}$$

which, combining with (1), yields

$$\mathring{C} = \delta, \quad \text{if} \quad K \notin \mathcal{A}.$$

Let

$$F(x, \theta) = \pi^{\alpha}(x) - \theta x, \quad x \geq 0, \theta > 0,$$

then, F satisfies all the conditions in the Lemma, and it is strictly concave in x, and hence, there exists a continuous function φ such that the unique solution of the optimization problem

$$\max_{x \geq 0} F(x, \theta)$$

is just $\varphi(\theta)$. Clearly, this $\varphi(\theta)$ is decreasing and converging to ∞ as $\theta \to 0$.

Therefore, noticing that $K(t)$ is just the unique solution of the optimization problem

$$\max_{x} \left\{ \pi^{\alpha}(x) + \lambda(t) \left(rZ(t) - x \right) \right\}.$$

and hence,

$$K(t) = \varphi(\lambda(t)),$$

it follows that $K(t)$ is continuous, so is $C(t)$.

In addition, by (4), we know that $\lambda(t)$ is strictly decreasing and converging to 0, and hence, $K(t)$ is increasing and converging to ∞, and $C(t)$ is increasing and converging to ∞, also.

Denote $K(0) = K_0$, $C(0) = C_0$. Then the range of $K(t)$ is $[K_0, \infty)$, and the range of $C(t)$ is $[C_0, \infty)$.

Suppose that in some time interval, it holds $K \in (a_{n-1}, a_n)$ for some $n \geq 1$, then,

$$\alpha C^{-\beta} k_{n-1} = \lambda,$$

then, there will be a time point, at which K touches a_n the first time, we denote this time point as T_n. K will stay in a_n for a while but not forever, and then, enter the interval (a_n, a_{n+1}), we denote the last time point, at which K stays in a_n as T_n'. Then, by continuity of $C(t)$,

$$\alpha C(T_n)^{-\beta} k_{n-1} = \lambda(T_n),$$

$$\alpha C(T_n')^{-\beta} k_n = \lambda(T_n'),$$

since $C(T_n) = C(T_n') = b^n$, then,

$$T_n' - T_n = \Delta_1 =: \frac{1}{r - \rho} \ln \frac{a}{b}.$$

Since

$$\alpha C(T_{n+1})^{-\beta} k_n = \lambda(T_{n+1}),$$

and $C(T_{n+1}) = b^{n+1}$, then,

$$T_{n+1} - T'_n = \Delta_2 =: \frac{\beta}{r-\rho} \ln b.$$

And hence, in general, the survival term of an industry is

$$\Delta =: \Delta_1 + 2\Delta_2 = \frac{1}{r-\rho} \ln a + \frac{\beta - \alpha}{r-\rho} \ln b.$$

Now, we determine K_0. By (3) and (4), we have

$$\lim_{t \to \infty} e^{-rt} Z(t) = 0,$$

it follows

$$Z(t) = \int_t^{\infty} e^{-r(s-t)} K(s) ds \geq K(t)/r, \tag{5}$$

which implies that $Z(t)$ is increasing and converging to ∞.

From (5), in particular, we get

$$Z_0 = \int_0^{\infty} e^{-rt} K dt, \tag{6}$$

which determines $K_0 \geq 0$ uniquely. We denote this relationship as $K_0 = g(Z_0)$.

We can prove (the process is trivial and awkward, omitted) that function g is continuous, piecewise smooth, and ladder-like such that on $[\Theta_n, \Theta'_n)$ for any n,

$$g \equiv a_n;$$

on $[\Theta'_n, \Theta_n)$ for any n, g is strictly increasing, where

$$0 = \Theta_0 < \Theta'_0 < \Theta_1 < \Theta'_1 < ... < \Theta_n < \Theta'_n < ...$$

are real numbers, depending only on α, ρ, r, a, b.

Furthermore, one can see that if $Z_0 \in [\Theta_{n_0}, \Theta'_{n_0})$ for some n_0, then, at the very beginning, there is only industry n_0, after a period of time (shorter than Δ_1), the economy will enter the next phase, where there are two industries: industry n_0 and industry $n_0 + 1$;......

If $Z_0 \in [\Theta'_{n_0}, \Theta_{n_0})$ for some n_0, then, at the very beginning, there are two industries: industry n_0 and industry n_0+1, after a period of time (shorter than Δ_2), the economy will enter the next phase, where there remains only industry $n_0 + 1$, ...

And correspondingly, in general, along with the increasing of the capital stock, the working capital is also increasing, and each industry $n > n_0$ experiences three phases: the rise phase, there are industry $n-1$ and industry n, the output of industry n increases from 0 to its peak 1, the time span of this phase is Δ_2; the peak phase, there is only industry n, its output remains at the peak 1, the time span of this phase is Δ_1; the descent phase, there are industry n and industry $n+1$, the output of industry n decreases from its peak 1 to 0. And hence, it demonstrates a trapezoid-like shape.

This is the pattern of the industry upgrading.

Remark 3. The relationship between the speed of industry upgrading (measured by $d =: 1/\Delta$) and the productivity parameter (b) depends on the intertemporal elasticity of substitution ($\tau =: 1/\beta$): if $\tau > 2$, then, d is increasing wrt b; if $\tau = 2$, then, d is independent of b; if $\tau < 2$, then, d is decreasing wrt b.

Since b represents the importance of high industries, and hence, intuitively, the higher it is, the faster the industries should upgrade. Therefore, if, in reality, $\tau > 2$, that will coincide with our analysis.

Remark 4. In general, in the survival term, the development of each industry follows a trapezoid-shaped pattern. The term of the peak period is increasing wrt to a/b, which could be quite long; the terms of rise and descent are equal and increasing wrt to b, which could be very short.

References

[1] Ju J, Lin J Y, Wang Y. Endowment structures, industrial dynamics, and economic growth. Journal of Monetary Economics, 2015, 76:244-263.