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Cobb-Douglas Technology Choice

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This paper studies how technology choice shapes aggregate production and structural transformation. We develop a general equilibrium framework in which all feasible technologies are Cobb–Douglas but differ in productivity and capital share. We show that the aggregate production function emerges as the envelope of firm-level choices, that any equilibrium can be represented with at most two technologies, and that equilibria may feature single-technology adoption, multi-technology mixtures, or non-existence depending on endowments and the technology set. In a multi-sector extension, development generates a predictable sequence of industry upgrading, with sectors shifting from labor-intensive to capital-intensive techniques along sharp phase boundaries. The framework provides a tractable foundation for linking micro-level technology menus to macroeconomic dynamics and offers new insights into the theory of structural change and long-run growth.

Keywords: Technology choice, Cobb-Douglas, aggregate production function, factor endowments, general equilibrium.

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Abstract. This paper studies how technology choice shapes aggregate production and structural transformation. We develop a general equilibrium framework in which all feasible technologies are Cobb–Douglas but differ in productivity and capital share. We show that the aggregate production function emerges as the envelope of firm-level choices, that any equilibrium can be represented with at most two technologies, and that equilibria may feature single-technology adoption, multi-technology mixtures, or non-existence depending on endowments and the technology set. In a multi-sector extension, development generates a predictable sequence of industry upgrading, with sectors shifting from labor-intensive to capital-intensive techniques along sharp phase boundaries. The framework provides a tractable foundation for linking micro-level technology menus to macroeconomic dynamics and offers new insights into the theory of structural change and long-run growth.

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1 Introduction

Technological change has been a central theme throughout economic history, and modern economies confront a rich *menu* of feasible technologies. At the micro level, firms select among technologies to minimize costs or maximize profits; at the macro level, such choices aggregate into economy-wide production possibilities that shape relative prices, the distribution of income between factors, and the trajectory of structural transformation. Understanding *which* technologies are chosen in equilibrium, and *how* those choices depend on factor endowments and preferences, is thus fundamental for both positive and normative macroeconomics.

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General equilibrium analysis can present panoramic view over the whole economy, and reveal the links between different factors. We now turn to discuss the issue of technology choice in general equilibrium framework. In the sequel, we only consider the Cobb-Douglas case.

This paper analyzes technology choice in general equilibrium under the maintained assumption that all feasible technologies are Cobb–Douglas (CD) with potentially different capital shares and TFP levels. Formally, we allow firms to choose from a set $\mathscr{F} = \{A(\alpha)K^{\alpha}L^{1-\alpha} : \alpha \in \Omega\}$, with $A(\alpha) > 0$ and $\Omega \subset [0, 1]$. The central questions are: (i) when does an *aggregate* production function exist as the envelope of firm-level choices, and when is it actually *realized* by a finite mixture of local technologies; (ii) how does the equilibrium technology (or mixture) vary with the economy's endowment ratio K/L; and (iii) in multi-sector economies, what governs the *order* and *timing* of industry-level upgrading?

Our approach proceeds in three steps. First, using a general representation of technology via transformation sets, we show that optimal technology choice in general equilibrium may be independent of factor endowments when the effective aggregate technology is already pinned down by primitives. Second, when technology is represented by production functions, we characterize global and aggregate production functions with convex-analytic tools, identifying when the aggregate coincides with the pointwise maximum of local technologies and when it becomes a linear segment spanned by a joint tangent. Third, by specializing to the CD family with variable $A(\alpha)$ and α, we obtain sharp, largely closed-form characterizations of equilibrium technology choice in both one-sector and multi-sector settings. A key insight is that—depending on the shape of $b(\alpha) = \ln[A(\alpha)\alpha^{\alpha}(1-\alpha)^{1-\alpha}]$—equilibria with interior or extreme technologies arise naturally, multi-technology mixtures occur only on well-defined endowment intervals, and non-existence is linked to failures of realizability when the optimal envelope cannot be attained.

Our main contributions are threefold. (i) With CD local technologies, the realized aggregate production can always be represented with at most two local technologies at any factor mix, implying that multi-technology equilibria occur only on endowment sets where envelope kinks bind. (ii) The slope and curvature of $b(\alpha)$ determine whether the optimal α lies in the interior (strict concavity), at extremes (convexity), or on two-point mixtures (joint tangencies). (iii) In multi-sector economies, as K/L rises, industries upgrade sequentially from low-α to high-α technologies, with clear phase boundaries determined by primitives $(A_i, \Omega_i, \theta_i)$.

The remainder of the paper is organized as follows. Section 2 presents a structured review of the literature. Section 3 formalizes the general technology-choice problem and develops properties of global and aggregate production functions. Section 4 embeds the analysis in general equilibrium and states equilibrium characterizations. Sections 5 and 6 analyze one-sector and multi-sector CD economies, including sequential upgrading and conditions for (non-)existence. Section 7 extends the framework to a dynamic setting. Section 8 concludes.

Beyond its technical contributions, the framework develops a tractable method for characterizing aggregate technologies from micro-level menus, offering a new foundation for the theory of technology choice in general equilibrium. It provides a unified theoretical lens through which technology choice, factor endowments, and structural transformation can be understood, extending the core foundations of growth and general equilibrium theory.

2 Literature review

The study of technology choice has undergone a long and evolving trajectory since the 1960s, when the debate over "appropriate technology" first emerged. Early discussions emphasized the idea that technologies developed in advanced economies may not be suitable for developing countries with very different factor endowments, wage structures, and institutional settings.

Schumacher's notion of intermediate technology [30] and Ahmed's reflections on development-oriented technological choice [8] stressed that adoption is constrained not only by technical feasibility but also by costs, skills, and the surrounding social environment. This line of thought shaped development practice for decades and emphasized the mismatch between frontier technologies and local needs. At the same time, economic theorists began to formalize the idea of induced innovation. Kennedy argued that the bias of technological progress is endogenously shaped by relative factor prices [24], while Drandakis and Phelps developed a formal model linking induced invention to growth dynamics and income distribution [13].

These early contributions were crucial in shifting the perception of technology from being an exogenous driver of growth to being a choice variable shaped by economic incentives. However, they largely remained at the level of conceptual arguments or partial-equilibrium analysis, and did not yet provide a consistent general equilibrium framework in which technology adoption, growth, and distribution could be analyzed together.

From the 1990s onward, appropriate technology became a central theme in modern growth and development economics. Basu and Weil showed that countries far from the world technology frontier tend to adopt techniques suited to their relative scarcities, and that such mismatches can explain persistent cross-country differences in productivity and income levels [10]. Hall and Jones documented large international differences in output per worker and argued that institutional and technological factors were key determinants [21]. Caselli and Coleman constructed formal models of the world technology frontier, showing how the menu of techniques available to different countries depends on their endowments and shapes their long-run growth paths [11].

Recently, Leon-Ledesma and Satchi incorporated appropriate technology into a balanced growth setting, ensuring compatibility between endowment-driven adoption and steady-state dynamics [27]. These studies offered a rich view of how adoption interacts with development. Yet their main limitation lies in the treatment of the technology set as largely exogenous: while they showed how endowments select among technologies, they did not explain how the shape of the aggregate production function emerges from the combination of micro-level menus.

A parallel stream of research advanced the theory of directed technical change. Acemoglu developed a series of seminal contributions showing that innovation is directed toward relatively abundant or expensive factors, and that institutional or policy variables can alter the bias of technological progress [1–7]. This framework explained why some economies experience capital-biased innovation while others experience labor-saving progress, and it illuminated the links between innovation incentives, income distribution, and growth. The directed technical change literature also expanded into areas such as environmental policy, where the direction of innovation is critical for the transition to clean technologies [6].

Empirical and theoretical studies, including Jones' analysis of the conditions under which aggregate production functions approximate CES [22], and Growiec's contributions on factor-augmenting progress and variable factor shares [14–20], deepened our understanding of how micro-level changes in technology map into aggregate regularities. This strand is highly influential, but its main limitation is its narrow focus on the innovation margin. While it powerfully explains the incentives to innovate in particular directions, it pays less attention to the adoption and selection of existing technologies, which is often more relevant in contexts where innovation is slow or external but adoption is pervasive.

In parallel, another body of work focused on factor-augmenting technical change and the behavior of factor shares. A long-standing empirical regularity is the approximate stability of the labor share over long horizons, a fact historically consistent with Cobb–Douglas production.

Yet more recent evidence indicates that labor shares are neither constant nor uniform across countries, industries, or time periods. Studies such as Growiec's microfoundations for CES functions [17], his exploration of factor-specific technology choice [18], and the joint work with Groth and McAdam on labor share cycles [19], emphasized the importance of explicitly modeling the choice of factor-augmenting techniques. Meta-analyses, such as Knoblach and Stöckl's review of substitution elasticities [25], provide empirical bounds for theoretical assumptions and highlight the heterogeneity across settings. This strand has the advantage of being closer to empirical regularities but often faces the limitation of requiring complex functional forms, which can obscure the transparency of the mechanisms at work.

In more recent years, the scope of technology choice has expanded beyond simple factor bias to include networks, supply chains, and other systemic interconnections. Acemoglu and Azar highlighted that firms' interconnections in production networks fundamentally shape the propagation of technology shocks and aggregate fluctuations [7]. Kopytov and coauthors extended this idea by incorporating supply chain uncertainty into models of endogenous production networks [26]. These studies illustrate that technology adoption is embedded not only in sectoral decisions but also in the architecture of interfirm linkages. This network perspective greatly enriches our understanding of aggregate dynamics but also comes with limitations: the resulting models are often highly complex, and the link to benchmark functional forms used in applied macroeconomics (such as Cobb–Douglas or CES) is less direct, making it harder to draw transparent policy implications.

Taken together, the literature shows a rich progression: from early concerns about appropriate technology and induced innovation, through models of frontier adoption and directed technical change, to contemporary perspectives on factor-augmenting choices, elasticities of substitution, and network propagation. Each stage added depth to our understanding, yet each left gaps: early work lacked general equilibrium rigor, later development models treated technology sets as exogenous, directed technical change overemphasized innovation while underplaying adoption, and network models offered complexity at the cost of tractability.

Against this background, the present paper focuses specifically on Cobb–Douglas (CD) technologies. The reason is twofold. First, CD remains analytically tractable and historically central: it provides a benchmark case where factor shares are constant, yet the aggregation of multiple CD techniques can generate non-trivial dynamics of shares, substitution, and structural change. Second, studying technology choice within the CD class clarifies the extent to which empirical regularities attributed to "Cobb–Douglas production" may in fact arise from an envelope of heterogeneous Leontief technologies selected according to endowments.

In this sense, the CD framework is not only a convenient simplification but also an economically meaningful laboratory for understanding how micro-level technology menus translate into aggregate outcomes. By addressing the limitations of earlier strands—whether their neglect of general equilibrium, their exclusive focus on innovation, or their abstraction from tractable forms—this paper positions CD technology choice as a natural and insightful focal point for advancing the theory of technology selection in macroeconomics.

3 General setup

Consider an economy with one representative individual and n industries, each with one representative firm. In any industry-i, the firm is allowed to take multiple technologies freely from a set of available technologies:

$$\mathscr{F}_i = \left\{ F_i^{(\alpha)} | \alpha \in \Omega_i \right\},$$

where $\Omega_i \subset [0,1]$, and

$$F_i^{(\alpha)}(K,L) = A_i(\alpha) K^\alpha L^{1-\alpha},$$

and A_i is a positive function, K, L are inputs of capital and labor, respectively. Denote the aggregate production function in sector i as F_i.

The representative individual has initial endowments of capital $K_0 > 0$ and labor

$L_0 > 0$, (let $k_0 := K_0/L_0$), and his utility function is

$$U(C_1, C_2, ..., C_n) = \prod_{i=1}^{n} C_i^{\theta_i},$$

where C_i is the consumption good-i, and $\theta_1, ..., \theta_n \in (0,1)$ are constants, satisfying $\sum_{i=1}^{n} \theta_i = 1$.

For simplicity, in the sequel, in any case, we use β to replace $1-\alpha$, γ to replace α/β, and with the same subscript and superscript as α.

Given the price system $(r, \omega, p_1, ..., p_n)$, where r, ω, p_i are the prices of capital, labor and the consumption good i, respectively. The firm-i's problem $(\mathbb{P}_i)$ is[1]

$$\begin{aligned} \max_{(\alpha_{ij}, K_{ij}, L_{ij})_{j=1,2}} \quad & \left\{ p_i \sum_{j=1}^{2} A_i(\alpha_{ij}) K_{ij}^{\alpha_{ij}} L_{ij}^{\beta_{ij}} - r \sum_{j=1}^{2} K_{ij} - \omega \sum_{j=1}^{2} L_{ij} \right\}, \\ \text{s.t.} \quad & \alpha_{ij} \in \Omega_i, \quad K_{ij} \geq 0, \quad L_{ij} \geq 0, \quad \forall i, j, \end{aligned}$$

and the individual's problem $(\mathbb{P})$ is

$$\begin{aligned} \max_{(C_1, ..., C_n)} \quad & U(C_1, ..., C_n) \\ \text{s.t.} \quad & \sum_{i=1}^{n} p_i C_i \leq r K_0 + \omega L_0, \end{aligned}$$

Definition 1. A nonnegative vector $\left(r, \omega, p_i, \alpha_{ij}^*, C_i^*, K_{ij}^*, L_{ij}^*\right)_{i=1,...,n, j=1,2}$ is called an equilibrium, if for any i, $(\alpha_{ij}^*, K_{ij}^*, L_{ij}^*)_{j=1,2}$ solves $(\mathbb{P}_i)$, and $(C_1^*, ..., C_n^*)$ solves $(\mathbb{P})$, and

$$C_i^* = \sum_{j=1}^{2} A_i(\alpha_{ij}^*) \left(K_{ij}^*\right)^{\alpha_{ij}^*} \left(L_{ij}^*\right)^{\beta_{ij}^*}, \quad \forall i,$$

$$\sum_{i=1}^{n} \sum_{j=1}^{2} K_{ij}^* = K_0, \quad \sum_{i=1}^{n} \sum_{j=1}^{2} L_{ij}^* = L_0.$$

For simplicity, due to the clearing condition for all the consumption goods markets, to express an equilibrium, we can omit writing out the C_i^*. And, if in any sector-i, only one technology is chosen, then we denote $(\alpha_{ij}, K_{ij}, L_{ij})_{j=1,2}$ simply as (α_i, K_i, L_i).

Definition 2. We say that two equilibria $(r, \omega, p_i, C_i, \alpha_{ij}, K_{ij}, L_{ij})_{i=1,...,n, j=1,2}$ and $\left(r, \omega, p_i, C_i, \alpha_{ij}', K_{ij}', L_{ij}'\right)_{i=1,...,n, j=1,2}$ are equivalent, if for any i,

$$\sum_j K_{ij} = \sum_j K_{ij}', \quad \sum_j L_{ij} = \sum_j L_{ij}'.$$

[1] By Theorem 1 in Appendix B, any firm takes at most two technologies simultaneously.

The equivalence means that the total output in each sector is the same, although the concrete technologies and the factor allocation within one sector are different.

For this economy, the social planner's problem is

$$\begin{aligned}
\max \quad & U(C_1, ..., C_n), \\
\text{s.t.} \quad & C_i = \sum_{j=1}^{2} A_i(\alpha_{ij}) K_{ij}^{\alpha_{ij}} L_{ij}^{\beta_{ij}}, \quad \forall i, \\
& \sum_{ij} K_{ij} = K_0, \quad \sum_{i,j} L_{ij} = L_0, \\
& \alpha_{ij} \in \Omega_i, \quad \forall i, j.
\end{aligned}$$

It's easy to see that for this economy, the first and second theorems of welfare economics hold. And hence, the equilibrium problem is equivalent to the social planner's problem (except for the price system), which is equivalent to identifying the realization of the aggregate production functions $(F_1, ..., F_n)$.

Define $B_i(\alpha) := A_i(\alpha)\alpha^\alpha \beta^\beta$, $\quad a_i(\alpha) := \ln A_i(\alpha)$, $\quad b_i(\alpha) := \ln B_i(\alpha)$, and for any $x > 0$,

$$\Lambda_i(x) := \arg\max_{\alpha \in \Omega_i} \{b_i(\alpha) - \alpha \ln x\}.$$

One can easily prove the following basic result on the equilibrium.

Lemma. $(r, \omega, p_i, \alpha_{ij}, K_{ij}, L_{ij})_{i=1,...,n, j=1,2}$ is an equilibrium, if and only if

$$\frac{\gamma_{ij} L_{ij}}{K_{ij}} = x, \quad \frac{r}{p_i} = B_i(\alpha_{ij}) x^{\beta_{ij}}, \quad \forall i, j;$$

$$\alpha_{ij} \in \Lambda_i(x), \quad \forall i, j;$$

$$x \sum_j K_{ij} + \sum_j L_{ij} = \theta_i (x K_0 + L_0), \quad \forall i;$$

$$K_0 = \sum_{ij} K_{ij}, \quad L_0 = \sum_{ij} L_{ij},$$

where $x = r/\omega$.

From this lemma, we can obtain following corollaries immediately.

Corollary 1. The equilibrium exists and is unique and is with one technology, if and only if $\Lambda_i(x) = \{\alpha_i\}$, $\forall i$, where

$$x = \frac{\sum_i \theta_i \alpha_i}{k_0 \sum_i \theta_i \beta_i}.$$

Corollary 2. If for any i, Ω_i is some interval in $[0, 1]$, and b_i is smooth and strictly concave. Then, the equilibrium exists and is unique and with one technology, and these technologies are all interior, if and only if the equations for $(x, \alpha_1, ..., \alpha_n)$, $x > 0, \alpha_i \in \Omega_i, i = 1, ..., n$:

$$\begin{cases} xk_0 = \sum_i \theta_i \alpha_i / \sum_i \theta_i \beta_i, \\ a_i'(\alpha_i) + \ln \gamma_i = \ln x, \quad \forall i = 1, ..., n \end{cases}$$

has a unique solution and the solution is interior in the sense that for any $i = 1, ..., n$, α_i is an interior point of Ω_i.

Corollary 3. If for any i, Ω_i is an closed interval in $[0, 1]$, and b_i is convex. Then, the unique (in the sense of equivalence) equilibrium (in case of existence) is the equilibrium taking with extreme technologies (maybe multiple), that is, for any i, the optimal α_i's are taken from the end points of Ω_i.

We see that the concavity/convexity of b_i matters.

We need to emphasize that in the above setting, the $A_i(\alpha)$'s are given exogenously. Which kind of $A_i(\alpha)$'s are suitable? We will discuss several typical types of it.

4 One-sector

Consider an economy stated in section 3 with $n = 1$. We use the same notations, except for that we drop the subscript i. Let the price of consumption good be 1.

4.1 The case with countable Ω

Suppose $\Omega = \{\alpha_n | n \in \mathbb{N}\}$, where $\{\alpha_n\}_{n \in \mathbb{Z}}$ is a sequence of strictly increasing numbers on $(0, 1)$. Here and throughout this paper, $\mathbb{N}$ denotes the set of all natural numbers.

Define $m_0 := \infty$, and for any $n \in \mathbb{N}$,

$$m_n := \frac{b(\alpha_{n+1}) - b(\alpha_n)}{\alpha_{n+1} - \alpha_n},$$

$$M_{n-} := \gamma_n e^{-m_{n-1}}, \quad M_{n+} := \gamma_n e^{-m_n}, \quad M^* := \sup_{n \in \mathbb{N}} M_{n+}.$$

Proposition 1. (i) If b is strictly concave, then, for $k_0 \in (0, M^*)$, the equilibrium exists and is unique, more precisely, for $k_0 \in (M_{n-}, M_{n+})$, the optimal technology is taking α_n; for any $k_0 \in [M_{n+}, M_{(n+1)-}]$, the optimal technology is taking α_n, α_{n+1} simultaneously; for $k_0 \geq M^*$, the equilibrium does not exist.

(ii) If for any $K_0 > 0, L_0 > 0$, the equilibrium exists and is unique, and there exists a sequence of real numbers $0 = M'_{1-} < M'_{1+} < ... < M'_{n-} < M'_{n+} < ...$ satisfying $\lim_{n \to \infty} M'_{n+} = \infty$ such that for $k_0 \in (M'_{n-}, M'_{n+})$, the optimal technology is taking α_n; for any $k_0 \in [M'_{n+}, M'_{(n+1)-}]$, the optimal technology is taking α_n, α_{n+1} simultaneously, then, b is strictly concave and $M'_{n-} = M_{n-}$, $M'_{n+} = M_{n+}$ for any $n \in \mathbb{N}$, and $M^* = \infty$.

Remark 1. When b is strictly concave, M_{n+} and $M_{(n+1)-}$ are the two tangent points of the joint tangent line of the two curves $y = A(\alpha_n)k^{\alpha_n}$ and $y = A(\alpha_{n+1})k^{\alpha_{n+1}}$ on the k-y plain.

Remark 2. If $M^* = \infty$, then, for any $K_0 > 0$, $L_0 > 0$, the equilibrium exists and is unique. Along with the increase of k_0 from 0 to ∞, the optimal α will go throughout $\alpha_1, \alpha_2, ...$ from left to right sequentially.

4.2 The case with $\Omega = [0, 1]$

Since the concavity of b matters, from Corollaries 2 and 3, we can get further corollaries in this case.

In particular, if b is smooth and strictly concave, then for any $K > 0, L > 0$, $F(K, L) = F^{(\alpha)}(K, L)$, where α is determined by $a'(\alpha) = \ln(L/K)$.

If b is convex, then, $F(K, L) = A(1)K + A(0)L, \quad \forall(K, L) \in \mathbb{R}^2_+$.

For more complicated case, one can verify that if a is strictly concave but b is strictly concave in $[0, \alpha_0]$ and strictly convex in $[\alpha_0, 1]$, where $\alpha_0 \in (0, 1)$[2], then, the equilibrium exists and is unique, and there are $k_* > 0$ and $\alpha_* \in (0, \alpha_0)$ such that the optimal technology is taking $\alpha \in (0, \alpha_*)$ and 1 simultaneously, the corresponding allocation of capital is $K_0 \wedge (k_* L_0)$ and $(K_0 - (k_* L_0))^+$, to the technologies α_* and 1, respectively, where α is determined by $a'(\alpha) = -\ln(K_0 \wedge (k_* L_0)$.

We now focus on the case, where b is linear, more precisely, $A(\alpha) = m^\alpha(\alpha^\alpha \beta^\beta)^{-1}$ for some constant $m > 0$. Then,

$$F(K, L) = A(\alpha_1)K_1^{\alpha_1} L_1^{\beta_1} + A(\alpha_2)K_2^{\alpha_2} L_2^{\beta_2},$$

for any $\alpha_i, K_i, L_i, i = 1, 2$ satisfying

$$K_1 + K_2 = K, \quad L_1 + L_2 = L, \quad mk_i = \gamma_i, i = 1, 2.$$

In other words, the equilibrium exists and is unique in the sense of equivalence class.

Among the equivalence class, two variants are special. One is the mixture of the two extreme technologies, that is, $\alpha = 1$ and $\alpha = 0$, which implies

$$F(K, L) = mK + L.$$

The other variant is one interior technology

$$\alpha^* = \frac{mK}{mK + L},$$

which yields

$$F(K, L) = A(\alpha^*)K^{\alpha^*} L^{\beta^*}.$$

[2]One example is $a(\alpha) = \alpha^\varepsilon$, where $\varepsilon \in (0, 1)$.

In addition, for this case, the TFP $A(\alpha)$ is inverted U-shaped in the interval $[0,1]$ and obtains its maximum at the point

$$\overline{\alpha} = \frac{m}{1+m},$$

which is approaching to 1 as $m \to \infty$. So, if m is sufficiently large, then, we see that in most part of the interval $[0,1]$, the TFP $A(\alpha)$ is strictly increasing.

To sum up, we get

Proposition 2. If b is linear, the optimal capital share strictly increases from 0 to 1, as the initial capital per capita k_0 increases from 0 to ∞.

5 Multi-sector

We discuss two cases separately.

5.1 Interior technology

Suppose that for any i, $\Omega_i = [0,1]$, and $A_i(\alpha) = \left(\alpha^\alpha \beta^\beta\right)^{-\delta_i}$, where $\delta_i > 1$, $i = 1, ..., n$, are constants.

For any i,

$$b_i(\alpha) = -(\delta_i - 1)(\alpha \ln \alpha + \beta \ln \beta)$$

is strictly concave, and hence, the equilibrium is with interior technologies.

More precisely, the equilibrium exits and is unique, in which for any i, the optimal technology for industry-i is

$$\alpha_i = \frac{1}{1 + x^{1/(\delta_i - 1)}},$$

where $x = r/\omega$, the rental-wage ratio, is determined uniquely by

$$k_0 x \sum_{i=1}^{n} \frac{\theta_i}{1 + x^{1/(1-\delta_i)}} = \sum_{i=1}^{n} \frac{\theta_i}{1 + x^{1/(\delta_i - 1)}}.$$

We see that x is strictly decreasing with respect to k_0; and hence, for any i, α_i is strictly increasing with respect to k_0.

Proposition 3. Along with the increase in k_0, each industry experiences technology upgrading, and the rental-wage ratio decreases continuously.

5.2 Extreme technology

Suppose that for any i, $\Omega_i = [\underline{\alpha}_i, \overline{\alpha}_i]$, and $A_i(\alpha) = m_i^\beta$, where $0 \le \underline{\alpha}_i < \overline{\alpha}_i \le 1$ and $m_i > 0$ are given constants, here m_i is the labor augmenting factor.

For any i,

$$b_i(\alpha) = \beta \ln m_i + \alpha \ln \alpha + \beta \ln \beta,$$

is strictly convex, and hence, the equilibrium must be with extreme technologies. That is, for any industry-i, the optimal technology is either low technology $\underline{\alpha}_i$; or high technology $\overline{\alpha}_i$; or mixed technology, that is, taking low and high technologies simultaneously.

For any $t \in \{1, ..., n\}$, define

$$\tau_t := \exp\left\{\frac{b_t(\underline{\alpha}_t) - b_t(\overline{\alpha}_t)}{\overline{\alpha}_t - \underline{\alpha}_t}\right\},$$

$$\kappa^*_{2t-1} := \frac{\sum_{j<t} \theta_j \overline{\alpha}_j + \sum_{j \geq t} \theta_j \underline{\alpha}_j}{\sum_{j<t} \theta_j \overline{\beta}_j + + \sum_{j \geq t} \theta_j \underline{\beta}_j} \tau_t, \quad \kappa^*_{2t} := \frac{\sum_{j \leq t} \theta_j \overline{\alpha}_j + \sum_{j>i} \theta_j \underline{\alpha}_j}{\sum_{j \leq t} \theta_j \overline{\beta}_j + \sum_{j>t} \theta_j \underline{\beta}_j} \tau_t,$$

and $\kappa^*_0 := 0$, $\kappa^*_{2n+1} := \infty$.

Assume $\tau_1 \leq \tau_2 \leq ... \leq \tau_n$. Then, $\kappa^*_0 \leq \kappa^*_1 \leq \kappa^*_2 \leq ... \leq \kappa^*_{2n} \leq \kappa^*_{2n+1}$.

Proposition 4. The equilibrium exits and is unique.

(i) If $k_0 \in [\kappa^*_{2t}, \kappa^*_{2t+1}]$ for some t, then,

$$\alpha_i = \overline{\alpha}_i, \quad \forall i \leq t; \qquad \alpha_i = \underline{\alpha}_i, \quad \forall i > t;$$

(ii) If $k_0 \in (\kappa^*_{2t-1}, \kappa^*_{2t})$ for some t, then,

$$\alpha_i = \overline{\alpha}_i, \quad \forall i < t; \qquad \alpha_{t1} = \underline{\alpha}_t, \quad \alpha_{t2} = \overline{\alpha}_t; \qquad \alpha_i = \underline{\alpha}_i, \quad \forall i > t.$$

Remark 3. In general, with the increase in k_0, any industry experiences technology upgrading through three phases, and the upgrading happens one by one sequentially. The total phases of economic development is at least $2n - 1$ (for the case, where $0 = \underline{\alpha}_i < \overline{\alpha}_i = 1$ for all i) and at most $2n + 1$ (for the case, where $0 < \underline{\alpha}_i < \overline{\alpha}_i < 1$ for all i).

In addition, Appendix C presents the analytical solution of the equilibrium, from which we can observe that the capital intensity of each sector increases with the rise in k_0.

Therefore, to summarize, we obtain the following result:

Proposition 5. (Technology Upgrading) As the per capita capital stock of the economy increases, each industry undergoes three phases of technological upgrading and becomes progressively more capital-intensive: low technology, mixed technology, and high technology. The upgrading process occurs sequentially—an industry begins upgrading only after the preceding one has completed its transformation. Consequently, the economy as a whole experiences a finite number of technological upgrading phases.

Remark 4. Analogously, one can show that if each firm is restricted to adopting a single technology, then an equilibrium fails to exist whenever $\kappa^*_{2t-1} < k_0 < \kappa^*_{2t}$ for some t.

6 The dynamic model

All the static models considered above can be extended to a dynamic one in this way: let all the industries in the static model be industries of intermediate goods, and introduce a new industry producing the final good (the unique consumption good, homogeneous with capital and hence can be accumulated as capital) by using all the intermediate goods as inputs through a production function as the individual's utility function in the static model, and set the individual's life-long utility as

$$\int_0^\infty e^{-\rho t} u(C) dt,$$

where C is his consumption of the final goods, $\rho > 0$ is his discount rate, and u is his instant utility function satisfying the usual conditions. In this way, we get a dynamic model, in which the capital is really strictly increasing in the whole process of economic development, and hence, the technology upgrading analyzed above can be applied in this dynamic setting. We omit the detail.

7 Conclusion

This paper develops a general equilibrium framework in which aggregate production arises from firms' choice among heterogeneous Cobb–Douglas technologies. We characterize when equilibria select a single technology, mix two technologies, or fail to exist, and we show that the curvature of the technology frontier governs these outcomes. In multi-sector settings, rising capital–labor ratios generate sequential industry upgrading along sharp phase boundaries. The framework provides a tractable foundation for linking micro-level technology menus to macroeconomic dynamics, offering new theoretical insights into structural transformation and long-run growth.

Beyond its theoretical contributions, the analysis also highlights mechanisms with direct policy relevance: as endowments evolve, economies traverse predictable paths of industrial upgrading, and policies that expand the feasible technology set or reduce adoption costs can accelerate this process. These insights help clarify the role of technology and industrial policy in shaping long-run development trajectories.

References

[1] Acemoglu, D. (2002). Directed technical change. *Review of Economic Studies*, 69, 782–809.

[2] Acemoglu, D. (2003). Labor- and capital-augmenting technical change. *Journal of the European Economic Association*, 1, 1–37.

[3] Acemoglu, D. (2007). Equilibrium bias of technology. *Econometrica*, 75(5), 1371–1410.

[4] Acemoglu, D., & Finkelstein, A. (2008). Input and technology choices in regulated industries: Evidence from healthcare. *Journal of Political Economy*, 116(5), 837–880.

[5] Acemoglu, D., Aghion, P., Bursztyn, L., & Hémous, D. (2012). The environment and directed technical change. *American Economic Review*, 102(1), 131–166.

[6] Acemoglu, D., Akcigit, U., Hanley, D., & Kerr, W. (2016). Transition to clean technology. *Journal of Political Economy*, 124(1), 52–104.

[7] Acemoglu, D., & Azar, P. D. (2020). Endogenous production networks. *Econometrica*, 88(1), 3–82.

[8] Ahmed, I. (1966). *Appropriate Technology: Choice and Development.* Macmillan, London.

[9] Arrow, K., Chenery, H. B., Minhas, B. S., & Solow, R. M. (1961). Capital-labor substitution and economic efficiency. *Review of Economics and Statistics*, 43, 225–250.

[10] Basu, S., & Weil, D. N. (1998). Appropriate technology and growth. *Quarterly Journal of Economics*, 113(4), 1025–1054.

[11] Caselli, F., & Coleman, W. J. (2006). The world technology frontier. *American Economic Review*, 96(3), 499–522.

[12] Chilarescu, C., & Vaneecloo, N. (2007). A stochastic approach to the Cobb–Douglas production function. *Economics Bulletin*, 3(9), 1–8.

[13] Drandakis, E. M., & Phelps, E. S. (1964). A model of induced invention, growth and distribution. *The Economic Journal*, 76(304), 823–840.

[14] Growiec, J. (2008). A new class of production functions and an argument against purely labor-augmenting technical change. *International Journal of Economic Theory*, 4, 483–502.

[15] Growiec, J. (2008). Production functions and distributions of unit factor productivities: Uncovering the link. *Economics Letters*, 101, 87–90.

[16] Growiec, J. (2013). Factor-augmenting technology choice and monopolistic competition. *Journal of Macroeconomics*, 38, 86–94.

[17] Growiec, J. (2013). A microfoundation for normalized CES production functions with factor-augmenting technical change. *Journal of Economic Dynamics and Control*, 37, 2336–2350.

[18] Growiec, J. (2018). Factor-specific technology choice. *Journal of Mathematical Economics*, 77, 1–14.

[19] Growiec, J., McAdam, P., & Mućk, J. (2018). Endogenous labor share cycles: Theory and evidence. *Journal of Economic Dynamics and Control*, 87, 74–93.

[20] Growiec, J. (2019). The hardware–software model: A new conceptual framework of production, R&D, and growth with AI. SGH KAE Working Paper 2019/042.

[21] Hall, R. E., & Jones, C. I. (1999). Why do some countries produce so much more output per worker than others? *Quarterly Journal of Economics*, 114(1), 83–116.

[22] Jones, C. I. (2005). The shape of aggregate production functions and the direction of technical change. *Quarterly Journal of Economics*, 120(2), 517–549.

[23] Ju, J., Lin, J. Y., & Wang, Y. (2015). Endowment structures, industrial dynamics, and economic growth. *Journal of Monetary Economics*, 76, 244–263.

[24] Kennedy, C. (1964). Induced bias in innovation and the theory of distribution. *The Economic Journal*, 74(295), 541–547.

[25] Knoblach, M., & Stöckl, F. (2020). What determines the elasticity of substitution between capital and labor? A literature review. *Journal of Economic Surveys*, 34(4), 847–875.

[26] Kopytov, A., Mishra, B., Nimark, K., & Taschereau-Dumouchel, M. (2024). Endogenous production networks under supply chain uncertainty. *Econometrica*, 92(5), 1621–1659.

[27] Leon-Ledesma, M. A., & Satchi, M. (2019). Appropriate technology and balanced growth. *Review of Economic Studies*, 86, 807–835.

[28] Lin, J. Y., Liu, Z., & Zhang, B. (2023). Endowment, technology choice, and industrial upgrading. *Structural Change and Economic Dynamics*, 65, 364–381.

[29] Rockafellar, R. T. (1970). *Convex Analysis*. Princeton University Press.

[30] Schumacher, E. F. (1962). *Appropriate Technology: An Intermediate Technology Development Group Paper*. Intermediate Technology Publications, London. Later developed into his 1973 book *Small is Beautiful*.

[31] Solow, R. M. (1957). Technical change and the aggregate production function. *Review of Economics and Statistics*, 39(3), 312–320.

[32] Uzawa, H. (1961). Neutral inventions and the stability of growth equilibrium. *Review of Economic Studies*, 28(2), 117–124.

[33] Zhang, B., Liu, Z., & Lin, J. Y. (2019). Endowments, technology choices and structural change. Institute of New Structural Economics, Peking University, Working Paper Series No. E2019005.

Appendix A. Transformation function

Technology can be expressed by a transformation function.

The idea "adapted technology" means that for any economy, technology choice should be adapted to its factor endowment and the people's preference but not its factor endowment only.

We now construct an example to show that the optimal technology choice may not depend on the economy's factor endowment.

Example. Consider an economy with one representative individual and one representative firm. Assume that the individual owns physical capital $K_0 > 0$ and labor $L_0 > 0$, and the firm produces two consumption goods using capital and labor by technologies chosen freely from the set of all available technologies:

$$\alpha x_1^2 + \beta x_2^2 \leq KL,$$

where K, L are the inputs of capital and labor, respectively, and x_1, x_2 are the outputs of good 1 and good 2, respectively; and α can be chosen in $[\alpha_1, \alpha_2]$, where $0 < \alpha_1 < \alpha_2 < 1$ are constants, and $\beta = 1 - \alpha$.

The individual's utility function is

$$U(C_1, C_2) = C_1^{\theta_1} C_2^{\theta_2},$$

where C_1, C_2 are his consumptions of the good 1 and good 2, respectively, and $\theta_1, \theta_2 \in (0,1)$ are constants, satisfying $\theta_1 + \theta_2 = 1$.

It is easy to verify that the equilibrium exists and is unique: the optimal technology choice is: taking technology α_1 only , if $\sigma_1 \leq 1$; taking technology α_2 only, if $\sigma_2 \leq 1$; taking technologies α_1, α_2 simultaneously, if $\sigma_i > 1$ for any $i = 1, 2$, where

$$\sigma_1 = \frac{\alpha_1}{\alpha_2}\theta_1 + \frac{\beta_1}{\beta_2}\theta_2, \quad \sigma_2 = \frac{\alpha_2}{\alpha_1}\theta_1 + \frac{\beta_2}{\beta_1}\theta_2.$$

Appendix B. Global and Aggregate Production Functions

In general, suppose that in a sector in an economy, the technology can be chosen freely from a set of available technologies:

$$\mathscr{F} = \left\{ F^{(\alpha)} | \alpha \in \Omega \right\},$$

where Ω is a set in some Euclidian space, and for any $\alpha \in \Omega$, $F^{(\alpha)}$ is a function of factor $X \in \mathbb{R}_+^n$, nonnegative, continuous, homogeneous, concave and increasing.

For any $N \in \mathbb{N}$, let

$$\begin{aligned} F_N(X) = \quad & \sup_{(\alpha_j, X_j)_{j=1,\ldots,N}} \quad \sum_{j=1}^{N} F^{(\alpha_j)}(X_j), \\ & \text{s.t.} \qquad X = \sum_{j=1}^{N} X_j, \\ & \qquad\qquad \alpha_j \in \Omega, \quad \forall j = 1, \ldots, N, \end{aligned}$$

and

$$G(X) = F_1(X); \quad F(X) = \sup_N F_N(X), \quad \forall X \in \mathbb{R}_+^n.$$

We call $F^{(\alpha)}$ a local production function for any $\alpha \in \Omega$, G global production function, and F aggregate production function.

The following basic result is a direct consequence of the classical Caratheodory theorem in convex analysis (See Corollary 17.1.3 in [29]).

Theorem 1. $F \equiv F_n$.

This theorem means that when the factor is of n-dimension, then, the aggregate production function can be realized by at most n local production functions at any concrete factor.

For a firm, facing the set of available technologies $\mathscr{F}$, the technology choice for it is to find F and its realization. The firm always takes F, if it can be realized.

For the global production function, we have the following result. Consider $\mathscr{F} = \left\{F^{(\alpha)}|\alpha \in \Omega\right\}$, where

$$\Omega = \left\{(a_1, ..., a_n) \in \mathbb{R}_+^n \middle| \prod_{i=1}^n a_i^{\theta_i} = 1\right\},$$

where $\theta_1, ..., \theta_n$ are all positive constants satisfying $\sum_{i=1}^n \theta_i = 1$; and for any $\alpha = (a_1, ..., a_n) \in \Omega$,

$$F^{(\alpha)}(X) = \Phi(a_1X_1, ...a_nX_n), \quad \forall X = (X_1, ..., X_n) \in \mathbb{R}_+^n,$$

where Φ, defined on $\mathbb{R}_+^n$, is smooth and satisfies that $\Phi(x) = 0$ for all x, which is not the interior point of $\mathbb{R}_+^n$, and increasing in the sense that Φ is increasing with respect to every element.

Then, the global production function is

$$G(X) = \max_{\alpha\in\Omega} F^{(\alpha)}(X), \quad \forall X \in \mathbb{R}_+^n.$$

Define a function ϕ on $[0, \infty)$:

$$\begin{aligned} \phi(z) = \quad & \max \quad \Phi(x_1, ..., x_n) \\ & \text{s.t.} \quad \prod_{i=1}^n x_i^{\theta_i} = z. \end{aligned}$$

We have a basic result. The proof is easy, hence, omitted.

Theorem 2.

(i) $G(X)$ exists for any $X \in \mathbb{R}_+^n$, if and only if $\phi(z)$ exists for any $z \in [0, \infty)$.

(ii) In case of existence, ϕ is increasing, and

$$G(X_1, ..., X_n) = \phi\left(\prod_{i=1}^n X_i^{\theta_i}\right), \quad \forall (X_1, ..., X_n) \in \mathbb{R}_+^n.$$

Therefore, G is homothetic.

(iii) In case of existence, G is smooth, if and only if ϕ is smooth. And, for any $X > 0$,

$$\frac{X_1G_1}{\theta_1} = ... = \frac{X_nG_n}{\theta_n},$$

where $G_i = \frac{\partial G}{\partial X_i}$, for any $i = 1, ..., n$.

(iv) If Φ is homogeneous, then, in case of existence, ϕ is homogeneous, and hence, G is itself Cobb-Douglas, and for any $i = 1, ..., n$, the share of the capital-i is θ_i.

Remark 5. If Φ is of CES type: for any $(x_1, ..., x_n) \in \mathbb{R}_+^n$,

$$\Phi(x_1, ..., x_n) = \left(\theta_1 x_1^\rho + ... + \theta_n x_n^\rho\right)^{1/\rho},$$

where $\rho < 1, \theta_i > 0, i = 1, ..., n$ are all constants, then, $\phi(z)$ exists for any $z \geq 0$, if and only if $\rho < 0$ (including the case $\rho = -\infty$).

Appendix C. Proofs

Proof of Proposition 1. We prove (i). First of all, we notice that an equilibrium with one technology exists, if and only if there exists $n \in \mathbb{N}$ such that $\Lambda(\gamma_n/k_0) = \{\alpha_n\}$, which is equivalent to

$$\varepsilon_n > \ln \frac{\gamma_n}{k_0} > \delta_n \tag{1}$$

where for any $n \in \mathbb{N}$,

$$\varepsilon_n = \min_{j<n} \frac{b(\alpha_j) - b(\alpha_n)}{\alpha_j - \alpha_n}, \quad \delta_n = \max_{j>n} \frac{b(\alpha_j) - b(\alpha_n)}{\alpha_j - \alpha_n},$$

in particular, $\varepsilon_1 = \infty$.

Since b is strictly concave, then, (1) is equivalent to

$$m_{n-1} > \ln \frac{\gamma_n}{k_0} > m_n, \tag{2}$$

where $m_0 = \infty$, and for any $n \in \mathbb{N}$,

$$m_n = \frac{b(\alpha_{n+1}) - b(\alpha_n)}{\alpha_{n+1} - \alpha_n}.$$

And, further, (2) is equivalent to

$$M_{n-} < k_0 < M_{n+},$$

where

$$M_{n-} = \gamma_n e^{-m_{n-1}}, \quad M_{n+} = \gamma_n e^{-m_n}.$$

We know

$$0 = M_{1-} < M_{1+} < M_{2-} < M_{2+} < \ldots < M_{n-} < M_{n+} < M_{(n+1)-} < M_{(n+1)+} < \ldots$$

Denote

$$M^* = \sup_{n\in\mathbb{N}} M_{n+}.$$

Obviously, $M^* = \infty$ if and only if $\alpha_n \to 1$ or $m_n \to -\infty$ as $n \to \infty$.

Now, suppose that there is $n \in \mathbb{N}$ such that

$$M_{n+} \leq k_0 \leq M_{(n+1)-} \tag{3}$$

According the above analysis, there does not exist an equilibrium with one technology. We turn to the equilibrium with multiple technologies.

Due to the strict concavity of b, it's easy to see that for any $x > 0$, $\Lambda(x)$ can not include more than two elements, and if it contains two elements, say, α_i, α_j, $i < j$, then, it must be the case, where $j = i + 1$.

So, under (3), if there is an equilibrium with two technologies, then, there is $x > 0$ and $i \in \mathbb{N}$ such that $\Lambda(x) = \{\alpha_i, \alpha_{i+1}\}$, and hence, $b(\alpha_i) - \alpha_i \ln x = b(\alpha_{i+1}) - \alpha_{i+1} \ln x$, then, $\ln x = m_i$. And, $x = \gamma_i/k_i = \gamma_{i+1}/k_{i+1}$; $L_0 = L_i + L_{i+1}$, $K_0 = K_i + K_{i+1}$. Therefore,

$$L_i = \frac{\gamma_{i+1} - xk_0}{\gamma_{i+1} - \gamma_i} L_0, \quad L_{i+1} = \frac{xk_0 - \gamma_i}{\gamma_{i+1} - \gamma_i} L_0,$$

$$K_i = \frac{\gamma_{i+1}/(xk_0) - 1}{\gamma_{i+1}/\gamma_i - 1} K_0, \quad K_{i+1} = \frac{1 - \gamma_i/(xk_0)}{1 - \gamma_i/\gamma_{i+1}} K_0.$$

Since all these four variables are nonnegative, then,

$$M_{i+} \le k_0 \le M_{(i+1)-}.$$

And hence, $i = n$.

Conversely, under (3), take $x = m_n$, and then, it's easy to verify that $\Lambda(x) = \{\alpha_n, \alpha_{n+1}\}$, and the equilibrium follows.

Thus, we can conclude that under (3), the equilibrium exists and is unique, which is the equilibrium with technologies n and $n+1$ simultaneously.

For any $k_0 \ge M^*$, there does not exist equilibrium. In fact, otherwise, there would exist $i \in \mathbb{N}$ such that $k_0 \le M_{(i+1)-} < M^*$. A contradiction. Therefore, the equilibrium does not exist.

To sum up, if $k_0 \in (0, M^*)$, then the equilibrium exists and is unique. Along with the increase of K_0 from 0 to M^*, the optimal α will go throughout $\alpha_1, \alpha_2, ...$ from left to right sequentially. If $k_0 \ge M^*$, then the equilibrium does not exist.

By the way, as to global production function, we have

$$F(K, L) = \begin{cases} F^{(\alpha_n)}(K, L), & \text{if} \quad K/L \in (M_{n-}, M_{n+}), \\ w_{n1}K + w_{n2}L, & \text{if} \quad K/L \in [M_{n+}, M_{(n+1)-}], \\ \text{not exist}, & \text{if} \quad K/L > M^*, \end{cases}$$

where $w_{n1} = B(\alpha_i)e^{m_n\beta_i}$, $w_{n2} = B(\alpha_2)e^{-m_n\alpha_i}$ for any $i = n, n+1$.

This completes the proof of (i). And (ii) can be proved by the same method, a bit modified.

Proof of Proposition 4. We only prove the result under the assumption that

$$0 < \underline{\alpha}_i < \overline{\alpha}_i < 1, \quad \forall i = 1, ..., n.$$

Other cases can be proved similarly.

Let $(r, \omega, p_i, C_{ij}, \alpha_{ij}, K_{ij}, L_{ij})_{i=1,...,n, j=1,2}$ be the equilibrium. We now consider the following two cases.

The first case. There exists some $t \in \{0, 1, ..., n\}$ such that for any $i \le t$, industry-i takes low technology; for any $i > t$, industry-i takes high technology, and no industry takes mixed technology.

By solving the individual's optimization problem, one can easily get that for any $i \in \{1, ..., n\}$,

$$p_i C_i = \theta_i (r K_0 + \omega L_0). \tag{4}$$

By solving the firms' optimization problems, one can get that for any $i \in \{1, ..., n\}$,

$$\frac{r}{p_i} = \alpha_i g_i^{-\beta_i}, \qquad \frac{\omega}{p_i} = m_i \beta_i g_i^{\alpha_i}, \tag{5}$$

where $g_i = K_i/(m_i L_i)$, and

$$\alpha_i = \begin{cases} \overline{\alpha}_i, & i \le t, \\ \underline{\alpha}_i, & i > t. \end{cases}$$

By the market clearing condition, we have

$$\sum_{i=1}^{n} K_i = K_0, \qquad \sum_{i=1}^{n} L_i = 1. \tag{6}$$

From (4),(5), we get that for any $i, j \in \{1, ..., n\}$,

$$\frac{\theta_i}{\theta_j} = \frac{p_i C_i}{p_j C_j} = \frac{\alpha_i^{-1} K_i}{\alpha_j^{-1} K_j} = \frac{\beta_i^{-1} L_i}{\beta_j^{-1} L_j},$$

which, combining with (6), gives that for any $i \in \{1, ..., n\}$,

$$K_i = \frac{\theta_i \alpha_i}{\sum_{j=1}^{n} \theta_j \alpha_j} K_0, \qquad L_i = \frac{\theta_i \beta_i}{\sum_{j=1}^{n} \theta_j \beta_j} L_0. \tag{7}$$

From (5)(7), we get that

$$\frac{\omega}{r} = \frac{\sum_{j \le t} \theta_j \overline{\beta}_j + \sum_{j > t} \theta_j \underline{\beta}_j}{\sum_{j \le t} \theta_j \overline{\alpha}_j + \sum_{j > t} \theta_j \underline{\alpha}_j} k_0. \tag{8}$$

And by solving the problems of the firms, we obtain

$$\left(\frac{\underline{\alpha}_i}{r}\right)^{\underline{\alpha}_i} \left(\frac{\underline{\beta}_i}{\omega/A_i}\right)^{\underline{\beta}_i} \le \left(\frac{\overline{\alpha}_i}{r}\right)^{\overline{\alpha}_i} \left(\frac{\overline{\beta}_i}{\omega/m_i}\right)^{\overline{\beta}_i}, \quad \forall i \le t;$$

$$\left(\frac{\underline{\alpha}_i}{r}\right)^{\underline{\alpha}_i} \left(\frac{\underline{\beta}_i}{\omega/m_i}\right)^{\underline{\beta}_i} \ge \left(\frac{\overline{\alpha}_i}{r}\right)^{\overline{\alpha}_i} \left(\frac{\overline{\beta}_i}{\omega/m_i}\right)^{\overline{\beta}_i}, \quad \forall i > t.$$

Therefore,

$$\tau_t \le \frac{\omega}{r} \le \tau_{t+1},$$

which, combining with (8), yields that

$$\kappa_{2t}^* \le k_0 \le \kappa_{2t+1}^*.$$

The second case. There exists some $t \in \{0, 1, ..., n\}$ such that for any $i < t$, industry-i takes low technology; for any $i > t$, industry-i takes high technology, and industry-t takes mixed technology.

By solving the individual's optimization problem, (4) still holds. By solving the firms' optimization problem, we obtain that for any $i \neq t$,

$$\frac{r}{p_i} = \alpha_i g_i^{-\beta_i}, \qquad \frac{\omega}{p_i} = m_i \beta_i g_i^{\alpha_i}, \tag{9}$$

where $g_i = K_i/(m_i L_i)$, and

$$\alpha_i = \begin{cases} \overline{\alpha}_i, & i < t, \\ \underline{\alpha}_i, & i > t, \end{cases}$$

and

$$\frac{r}{p_t} = \underline{\alpha}_t g_{t1}^{-\underline{\beta}_t} = \overline{\alpha}_t g_{t2}^{-\overline{\beta}_t}, \quad \frac{\omega}{p_t} = m_t \underline{\beta}_t g_{t1}^{\underline{\alpha}_t} = m_t \overline{\beta}_t g_{t2}^{\overline{\alpha}_t}, \tag{10}$$

where $g_{tj} = K_{tj}/(m_t L_{tj})$. And hence,

$$\frac{\omega}{r} = \frac{\overline{\beta}_t}{\overline{\alpha}_t} k_{t2} = \frac{\underline{\beta}_t}{\underline{\alpha}_t} k_{t1}, \qquad \overline{\beta}_t g_{t2}^{\overline{\alpha}_t} = \underline{\beta}_t g_{t1}^{\underline{\alpha}_t},$$

which yields

$$\frac{\omega}{r} = \tau_t.$$

By (4) and (9), we have that for any $i \neq t$,

$$K_i = \theta_i \alpha_i (K_0 + \tau_t L_0), \qquad L_i = \theta_i \beta_i (K_0/\tau_t + L_0),$$

and by (4) and (10), we obtain

$$\frac{K_{t1}}{\underline{\alpha}_t} + \frac{K_{t2}}{\overline{\alpha}_t} = \theta_t (K_0 + \tau_t L_0), \qquad \frac{L_{t1}}{\underline{\beta}_t} + \frac{L_{t2}}{\overline{\beta}_t} = \theta_t (K_0/\tau_t + L_0).$$

By the market clearing condition, we have

$$K_{t1} + K_{t2} = K_0 - \sum_{i \neq t} \theta_i \alpha_i (K_0 + \tau_t L_0); \qquad L_{t1} + L_{t2} = 1 - \sum_{i \neq t} \theta_i \beta_i (K_0/\tau_t + L_0).$$

Therefore,

$$K_{t1} = \frac{\sum_{j \le t} \theta_j \overline{\beta}_j + \sum_{j > t} \theta_j \underline{\beta}_j}{\overline{\alpha}_t/\underline{\alpha}_t - 1} \left(\frac{\kappa_{2t}^*}{k_0} - 1 \right) K_0,$$

$$K_{t2} = \frac{\sum_{j < t} \theta_j \overline{\beta}_j + \sum_{j \ge t} \theta_j \underline{\beta}_j}{1 - \underline{\alpha}_t/\overline{\alpha}_t} \left(1 - \frac{\kappa_{2t-1}^*}{k_0} \right) K_0,$$

$$L_{t1} = \frac{\sum_{j \le t} \theta_j \overline{\alpha}_j + \sum_{j > t} \theta_j \underline{\alpha}_j}{1 - \overline{\beta}_t/\underline{\beta}_t} \left(1 - \frac{k_0}{\kappa_{2t}^*} \right) L_0,$$

$$L_{t2} = \frac{\sum_{j<t} \theta_j \overline{\alpha}_j + \sum_{j \geq t} \theta_j \underline{\alpha}_j}{\underline{\beta}_t / \overline{\beta}_t - 1} \left(\frac{k_0}{\kappa^*_{2t-1}} - 1 \right) L_0.$$

And, obviously, in this case,

$$\kappa^*_{2t-1} \leq k_0 \leq \kappa^*_{2t}.$$

By the same method, one can prove that any other cases are impossible. The proof is completed.